

$SU(2)$ particle sigma model: the role of non-point symmetries in global quantization

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Abstract

In this paper we achieve the quantization of a particle moving on the $SU(2)$ group manifold, that is, the three-dimensional sphere S^3 , by using group-theoretical methods. For this purpose, a fundamental role is played by contact, non-point symmetries, i.e., symmetries that leave the Poincaré-Cartan form semi-invariant at the classical level, although not necessarily the Lagrangian. Special attention is paid to the role played by the basic quantum commutators, which depart from the canonical, Heisenberg-Weyl ones, as well as the relationship between the integration measure in the Hilbert space of the system and the non-trivial topology of the configuration space. Also, the quantization on momentum space is briefly outlined.

Keywords: Nonlinear systems, sigma model, contact symmetries, non-canonical quantization, non trivial topology.

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1 Introduction

One of the main problems which faces theoretical physics today is the proper quantization of non-linear dynamical systems evolving on a configuration space with non-trivial topology. In fact, the difficulties encountered in performing the quantization of paradigmatic non-linear systems, like gravitational fields, is traditionally associated with inherent incompatibilities between Quantum Mechanics and the corresponding interaction. The only true assertion in this respect seems to be that Canonical Quantization is incompatible with non-linearity in general. But this is a very well-known fact established at the earliest stage of Quantum Mechanics through the No-Go theorems [1] (see [2] for a review).

Since the very beginning, symmetry principles have constituted a helpful tool in describing quantum phenomena [3], although they were focussed mostly on the reconstruction of new solutions related to a given one by symmetry transformations. A deeper use of symmetries were made in more recent approaches to quantization, like Geometric Quantization [4, 5], based on an extension to Physics of the geometric technique of group representation on co-adjoint orbits of Lie Groups [6]. However, those geometric methods found serious limitations concerning the type of operators which were compatible with the process of reduction of the quantum representation, that is, the “polarization” of the wave function.

In this paper we face the quantum description of an intrinsically non-linear system with configuration space bearing a non-trivial topology and, nevertheless, simple enough so as to be solved exactly. This is a highly symmetric system, that of a free particle evolving on the sphere S^3 , which will allow us to employ a Group Approach to Quantization (see [7] and references therein; see especially the pioneer papers on this subject [8, 9]) improving the mentioned Kostant-Kirillov-Souriau geometric method. Taking as starting point the group of generalized symmetries (in the sense that it includes non-point symmetries of the Poincaré-Cartan form) of our system and, using intrinsic geometric and algebraic structures of Lie groups, we shall construct consistently and unambiguously the unitary and irreducible representations of the basic symmetry group, which will constitute eventually the possible quantum representations of the physical system. This method has proven very efficient in several non-trivial (even infinite-dimensional) systems; a significant example was the quantization of all orbits of the Virasoro group [10], appearing in 2D quantum Gravity. The interest of the present example goes beyond the mere study of the quantum dynamics of a particle on a Riemannian manifold; it will constitute a toy model for the highly non-trivial task of properly quantizing Non-Linear Sigma Models in field theories and, in particular, the boson sector of massive Yang-Mills theories. In fact, the (Lie algebra of the) local version of the relevant symmetry here encountered, to be named, local Sigma $SU(2)$ group, was already pointed out in [11].

The paper is organized as follows. In Sec. 2 we describe the basic symmetries of the classical system of a particle moving on a Riemannian manifold paying special attention to the non-point character of part of them, precisely those generalizing the boosts in the case of the flat geometry. The particular case of the free particle moving on S^3 is considered in Subsec. 2.2. In Sec. 3 we perform the quantization of the Particle-Non-Linear-Sigma-Model associated with $SU(2)$, that is, the free particle moving on the $SU(2)$ group manifold as configuration space, according to a non-canonical Group Approach to Quantization, briefly introduced in Subsec 3.1. The complete quantization is performed in configuration space, and the eigenvectors of the Hamiltonian operator, which turns out to be the Laplace-Beltrami operator for S^3 , are computed. The realization in momentum space is also briefly discussed. Sec 4 is devoted to remarking different non-trivial aspects which appear as related to both non-trivial topology and non-canonical basic commutation relations.

2 Classical description and Symmetries

We are interested in fundamental (elementary) systems defined by an action. The relevant symmetries for classical mechanics are the Noether-contact symmetries, i.e., those which leave semi-invariant the Poincaré-Cartan form of the system [12]. So often it is not noticed that in the proof of her theorem, Noether considered symmetries beyond point symmetries [13]. One of our objectives in this article is to show the importance and necessity of incorporating, naturally, non-point symmetries at the classical and quantum level.

The particle moving on the sphere S^3 is an example of what is known in physics by a particle sigma model. Our interest in this model lies in two aspects that make it particularly relevant. Firstly, it is a typical non-linear model enclosing the main problems that may be relevant to quantum mechanics and, secondly, the topology of the configuration space is non-trivial.

2.1 Lagrangian formalism on Riemannian manifolds

The general particle nonlinear sigma model (PNLσm) over a Riemann manifold (the target manifold) is defined, traditionally, by the following action,

$$S = \int dt L = \int dt \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j, \quad (1)$$

where L is the Lagrangian, $\dot{x}^i = \frac{dx^i}{dt}$, t is the parameter of the curve, $g_{ij}(x)$ is the metric of the manifold and, as usual, the connection on the manifold is the Levi-Civita connection (Christoffel

symbols). From this action, and according to the Ordinary Hamilton Principle (OHP), we derive the Euler-Lagrange equations of motion:

$$\dot{x}^i \nabla_i \dot{x}^j \equiv \ddot{x}^j + \Gamma_{kl}^j \dot{x}^k \dot{x}^l = 0, \quad (2)$$

where ∇_i is the covariant derivative. These equations are the geodesics equations of the manifold. The canonical momentum and the Hamiltonian function are

$$\frac{\partial L}{\partial \dot{x}^i} = p_i = g_{ij} \dot{x}^j, \quad (3)$$

and

$$H = \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L = \frac{1}{2} g^{ij} p_i p_j = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} |\dot{x}|^2. \quad (4)$$

Obviously H is a constant of motion.

Also in the framework of the OHP, the symmetries of any system defined by a Lagrangian are obtained by imposing the condition of invariance of the action, which in this case corresponds with the semi-invariance of the action integrand, that is, invariance up to a total derivative. If X is the generator of a one-parameter group of transformations of the space (t, x^i, \dot{x}^j)

$$X = X^t \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i} + \dot{X}^i \frac{\partial}{\partial \dot{x}^i}. \quad (5)$$

then this condition is

$$L_X L + L \frac{dX}{dt} = \frac{df_X}{dt}, \quad (6)$$

where L_X indicates the Lie derivative with respect to the vector field X , the operator $\frac{d}{dt}$ is a total derivative and f_X is some function which depends on X .

The explicit form of the generator depends on the type of symmetry that we are looking for. Usually, and again in the framework of the ordinary variational calculus, the components $X^t(t)$, $X^i(x, t)$ correspond to the infinitesimal version of a transformation on x^i and perhaps the evolution parameter t :

$$t' = t + \tau(t), \quad x'^i = x^i + \xi^i(x, t)$$

whereas the component \dot{X}^i is the infinitesimal action on \dot{x}^i induced from those of x^i and t , that is:

$$\dot{X}^i = \frac{\partial X^i}{\partial t} + \frac{\partial X^i}{\partial x^k} \dot{x}^k - \frac{\partial X^t}{\partial t} \dot{x}^i \quad (7)$$

A particularly interesting case is that of geometric transformations where the parameter t is not transformed and ξ does not depend on t so that \dot{X}^i is reduced to

$$\dot{X}^i = \frac{\partial X^i}{\partial x^k} \dot{x}^k.$$

Now the Lie derivative of L with respect to X results in

$$L_X L = X^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i} g_{kj} + \frac{\partial X^k}{\partial x^j} g_{ik} = \frac{1}{2} (L_X g_{ij}) \dot{x}^i \dot{x}^j, \quad (8)$$

where by $L_X g_{ij}$ we mean the traditional expression in Riemannian geometry

$$L_X g_{ij} = X^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i} g_{kj} + \frac{\partial X^k}{\partial x^j} g_{ik} \quad (9)$$

The invariance condition thus reads

$$L_X g_{ij} = 0 = \nabla_i X_j + \nabla_j X_i. \quad (10)$$

These equations are known as the Killing equations and the vectors $X^i(x)$ as the Killing vectors. This shows that the isometries of the metric are particular symmetries of the associated Lagrangian system. Therefore, it is then manifest that all sigma models have, at least, these t -independent point (geometric) symmetries.

Evidently, isometries are not the only point symmetries we could have considered. Symmetries (5) with the infinitesimal action on \dot{x}^i given by (7) are referred to as point symmetries and we say that (5) is the jet prolongation of the generator of the action on just t, x^i , $X^t \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i}$ [12, 13, 14]. Killing symmetries are then point symmetries independent of t and preserving t .

Point symmetries do not exhaust, however, all symmetries required to parameterize the set of solutions of the system described by (1), by means of the corresponding Noether invariants. More general, contact, non-point symmetries are required (save for the linear case corresponding to the trivial metric). They are defined, in the framework of the Modified Hamilton Principle (MHP) [15, 12] as those generated by vector fields with the general form (5), although with general \dot{X}^i (not satisfying (7)) leaving (semi-)invariant the Poincaré-Cartan form

$$\Theta_{PC} = \frac{\partial L}{\partial \dot{x}^i} (dx^i - \dot{x}^i dt) + L dt = g_{ij} \dot{x}^j dx^i - \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j d\tau = p_i dx^i - H dt. \quad (11)$$

Contact symmetries provide those symmetries generalizing the “boost” of the trivial-metric system, that is, free particle in Euclidean space (with Noether invariant the initial “position”). We could say that Killing symmetries only provide the generalized momenta but generalized position are, in general, of non-point character and have been traditionally hidden somehow.

2.2 Particle on the $SU(2)$ group manifold: S^3

One example containing all the ingredients of PNL σ M (in spaces of constant curvature) is that of a particle moving on a Lie group manifold G . In general we would be interested in semi-simple groups. For these groups we can define a two-side (chiral) invariant metric, i.e., invariant by the left and right action of the group, as follows [16, 17]:

$$g_{ij} = k_{ab} \theta_i^{(a)} \theta_j^{(b)}; \quad a, b, i, j = 1, \dots, \dim(G), \quad (12)$$

where $k_{ab} = C_{aj}^i C_{bi}^j$ is the Cartan-Killing metric (for semi-simple algebras this metric is nonsingular), C_{kj}^i are the structure constants of the Lie algebra of the group, and $\theta_i^{(a)}$ the left/right invariant 1-forms of Cartan¹. The Lagrangian driving the motion on G is given by

$$L = \frac{1}{2} m k_{ab} \theta_i^{L(a)} \theta_j^{L(b)} \dot{g}^i \dot{g}^j = \{L \rightarrow R\}, \quad (13)$$

where g^i are local coordinates of the group and m is the mass of the particle. As a general fact, the equations of motion can be written as

$$\frac{d}{dt} \theta_i^{L(a)} \dot{g}^i \equiv \frac{d}{dt} \theta^{La} = 0.$$

Here, we are particularly interested in the case in which G is the group $SU(2)$, the universal covering of the ordinary rotation group $SO(3)$. Thus, the configuration space of the particle is S^3 and the phase space is its tangent space $T(S^3)$. We shall parameterize locally the group manifold with coordinates $\{\epsilon^i; i = 1, 2, 3\}$, where $\vec{\epsilon}$ determines the axis of rotation and $|\vec{\epsilon}| = R \sin \frac{\varphi}{2}$, φ being the corresponding rotation angle; we make explicit the radius R of the sphere where the particle evolves. In fact, when completed with $R\rho(\vec{x}) \equiv R\sqrt{1 - \frac{\vec{x}^2}{R^2}}$, these coordinates are the restriction of Cartesian coordinates on a \mathbb{R}^4 Euclidean space to the three-dimensional sphere of radius R . With this parametrization we can derive the right- and left-invariant canonical 1-forms,

$$\theta^{R(i)} = \theta_j^{R(i)} d\epsilon^j = \left(\rho(\vec{\epsilon}) \delta_j^i + \frac{\epsilon^i \epsilon_j}{R^2 \rho(\vec{\epsilon})} + \frac{1}{R} \eta_{jk}^i \epsilon^k \right) d\epsilon^j, \quad (14)$$

¹Notice that the canonical 1-forms are a particular case of vierbeins $e_i^{(a)}$ as defined in [18]

$$\theta^{L(i)} = \theta_j^{L(i)} d\epsilon^j = \left(\rho(\vec{\epsilon}) \delta_j^i + \frac{\epsilon^i \epsilon_j}{R^2 \rho(\vec{\epsilon})} - \frac{1}{R} \eta_{jk}^i \epsilon^k \right) d\epsilon^j, \quad (15)$$

where the function ρ is defined above, η_{jk}^i is the Levi-Civita symbol in 3-dimensions, and the Cartan-Killing metric, or chiral metric, on the group

$$k_{ij} = \frac{2}{R} \eta_{im}^l \frac{2}{R} \eta_{jl}^m = -\frac{8}{R^2} \delta_{ij}, \quad (16)$$

$$-\frac{R^2}{8} k_{kl} \theta_i^{R(k)} \theta_j^{R(l)} = g_{ij} = \left(\delta_{ij} + \frac{\epsilon_i \epsilon_j}{R^2 \rho(\vec{\epsilon})^2} \right). \quad (17)$$

The dual vectors

$$Z_{(i)}^R = Z_{(i)}^{Rk} \frac{\partial}{\partial \epsilon^k} = \left(\rho(\vec{\epsilon}) \delta_i^k + \frac{1}{R} \eta_{ij}^k \epsilon^j \right) \frac{\partial}{\partial \epsilon^k}, \quad (18)$$

$$Z_{(i)}^L = Z_{(i)}^{Lk} \frac{\partial}{\partial \epsilon^k} = \left(\rho(\vec{\epsilon}) \delta_i^k - \frac{1}{R} \eta_{ij}^k \epsilon^j \right) \frac{\partial}{\partial \epsilon^k}, \quad (19)$$

constitute the infinitesimal generators of the left and right action of the group on itself, respectively, and will turn out to be the Killing vectors for the Cartan-Killing metric (see below). The Lie algebra (of the right-invariant vector fields) for $SU(2)$ is

$$\left[Z_{(i)}^R, Z_{(j)}^R \right] = -\frac{2}{R} \eta_{ij}^k Z_{(k)}^R. \quad (20)$$

From the formulas above, we write the Lagrangian for the S^3 PNL σ M:

$$L = \frac{1}{2} m g_{ij} \dot{\epsilon}^i \dot{\epsilon}^j = \frac{1}{2} m \left(\delta_{ij} + \frac{\epsilon_i \epsilon_j}{R^2 \rho(\vec{\epsilon})^2} \right) \dot{\epsilon}^i \dot{\epsilon}^j \quad (\dot{\epsilon}^i = \frac{d\epsilon^i}{dt}). \quad (21)$$

Note that this expression can also be obtained by constraining a 4D Lagrangian to the sphere S^3 . The Poincaré-Cartan form, the momentum and the Hamiltonian are [12]

$$\Theta_{PC} = \frac{\partial L}{\partial \dot{\epsilon}^i} (d\epsilon^i - \dot{\epsilon}^i dt) + L dt = p_i d\epsilon^i - H dt, \quad (22)$$

$$p_i = \frac{\partial L}{\partial \dot{\epsilon}^i} = m g_{ij} \dot{\epsilon}^j = m \theta_i^{R(k)} \theta_k^R, \quad (23)$$

$$H = \frac{\partial L}{\partial \dot{\epsilon}^i} \dot{\epsilon}^i - L = \frac{1}{2} m g_{ij} \dot{\epsilon}^i \dot{\epsilon}^j = \frac{1}{2m} g^{-1ij} p_i p_j = \frac{1}{2} m \delta_{ij} \theta^{Ri} \theta^{Rj}, \quad (24)$$

where $g^{-1ij} = \delta^{ij} - \frac{1}{R^2} \epsilon^i \epsilon^j$ is the inverse of the metric and we have called $\theta^i \equiv \theta_k^{R(i)} \epsilon^k$ (from now on, when the script R, L is omitted we understand R). We would then write:

$$\Theta_{PC} = m \theta_i \theta^{(i)} - \frac{1}{2} m \theta_i \theta^i dt.$$

The solutions of the equations of motion are

$$\begin{aligned} \epsilon^i &= \epsilon^i \cos \omega t + \dot{\epsilon}^i \frac{\sin \omega t}{\omega} \\ \dot{\epsilon}^i &= \dot{\epsilon}^i \cos \omega t - \omega \epsilon^i \sin \omega t, \end{aligned} \quad (25)$$

where $\omega = \sqrt{\frac{8}{mR^2} H} = \frac{1}{R} \sqrt{g_{ij}(\vec{\epsilon}) \dot{\epsilon}^i \dot{\epsilon}^j} = \frac{1}{R} \sqrt{g_{ij}(\vec{\epsilon}) \dot{\epsilon}^i \dot{\epsilon}^j}$; $\vec{\epsilon}$ and $\dot{\vec{\epsilon}}$ denote initial values for $\vec{\epsilon}(t)$ and $\dot{\vec{\epsilon}}(t)$. It must be noticed that these equations are similar to those of a harmonic oscillator but with a frequency depending on the energy. Note that the second equation is equivalently written as

$$\theta^i(\vec{\epsilon}(t)) = \text{const} \equiv \vartheta^i(\vec{\epsilon}) \quad (\text{either Left- or Right-}) \quad (26)$$

where $\vartheta^i(\vec{\varepsilon})$ keeps the same functional dependence on $\vec{\varepsilon}$ as θ^i on $\vec{\epsilon}$.

The expression (25), completed with the trivial one $t = \tau$, can be seen as an invertible transformation which goes from variables $(t, \epsilon^i, \dot{\epsilon}^j)$ to “constant” coordinates and velocities $(\tau, \varepsilon^i, \dot{\varepsilon}^j)$, in much the same way, in the case of the free particle, the expressions $t = \tau$, $x^i = x_0^i + \dot{x}_0^i t$, $\dot{x}^i = \dot{x}_0^i$ can be given the same interpretation. We shall call the mentioned transformation the “Hamilton-Jacobi” transformation, taking the name from the canonical transformation that take a Hamiltonian system to canonical constant coordinates and momenta (and null Hamiltonian) [19]. We refer the reader to Figure 1 in the Appendix to visualize the transformation and the notation used for the coordinates on each space.

Equation (26) points the natural constants of motion associated to the (Killing) symmetries (18) (or (19)), when jet-prolonged to a point symmetry (according to the general formula (7)), of the Lagrangian, that is

$$X_{(i)} = Z_{(i)}^j \frac{\partial}{\partial \epsilon^j} + \frac{\partial Z_{(i)}^j}{\partial \epsilon^k} \dot{\epsilon}^k \frac{\partial}{\partial \dot{\epsilon}^j} = Z_{(i)}^j \frac{\partial}{\partial \epsilon^j} + \frac{2}{R} \eta_{.in}^k \theta^n \frac{\partial}{\partial \theta^k}. \quad (27)$$

Even more, the use of θ^i instead of $\dot{\epsilon}^i$ has the advantage that the former is an intrinsic quantity associated with the group $SU(2)$ irrespective of the particular, local parametrization of the group (the configuration space).

A less obvious matter, however, is to realize the constants ε^i also as Noether invariants so that we can have the Solution Manifold (SM) parameterized by intrinsic quantities. To achieve this task, let us perform the Hamilton-Jacobi transformation on the Poincaré-Cartan form (22), that is, (25) completed with the trivial transformation $t = \tau$. After some computations we arrive at

$$\Theta_{PC} \approx \Lambda \equiv m \vartheta_i \vartheta^{(i)} \quad (28)$$

up to a total differential (note that ϑ_i is a function whereas $\vartheta^{(i)}$ is a 1-form, the canonical 1-form). The differential $d\Theta_{PC}$ does, actually, go to SM (that is to say, it can be written in terms of functions on SM only), defining the symplectic structure Ω , generated by the Liouville 1-form Λ :

$$\Omega = d\Lambda = m d\vartheta_i \wedge \vartheta^{(i)} + \frac{m}{R} \vartheta_i \eta_{.jk}^i \vartheta^{(j)} \wedge \vartheta^{(k)}. \quad (29)$$

The formulas above are written in intrinsic coordinates so that those expressions are valid for any local parametrization of the configuration space, S^3 , as a consequence of its being a Lie group. Nevertheless, we may resort to a particular useful coordinates, Darboux local coordinates, where (28) and (29) adopt their canonical form:

$$\begin{aligned} \Lambda &= \pi_i d\varepsilon^i \\ \Omega &= d\pi_i \wedge d\varepsilon^i \end{aligned} \quad (30)$$

where π_i is defined by $m\vartheta_i \equiv Z_{(i)}^k \pi_k$ or $\pi_i \equiv m\vartheta_i^{(k)} \vartheta_k$.

We are now in conditions to find out the symmetries of the classical system whose Noether invariants are the constant of motion ε^i . In fact, the only thing which remains is to compute the Hamiltonian vector field on SM, $Y^{(i)}$, associated with the Hamiltonian function ε^i according to the symplectic form (29):

$$Y^{(i)} = \frac{\partial}{\partial \pi_i} = \frac{1}{m} Z(\vec{\varepsilon})_{(k)}^i \frac{\partial}{\partial \vartheta_k}, \quad \Omega(Y^{(i)}) = d\varepsilon^i \quad (31)$$

To express these vectors, here obtained on SM, as symmetries of the original Poincaré-Cartan form (not a symmetry of the Lagrangian, though) we must perform the inverse Hamilton-Jacobi transformation, that is, the inverse of (25), completed again with the trivial one $\tau = t$. After some rather involved calculations, we arrive at

$$\begin{aligned} Y_{(j)} &= \frac{1}{m\omega} \left((\delta_j^n - \frac{1}{R^2} \epsilon_j \epsilon^n - \frac{\theta^k \theta^s}{m^2 R^2 \omega^4} Z_{(k)j} Z_{(s)}^n) \sin(\omega t) + \frac{\theta^l \theta^r t}{m^2 R^2 \omega} Z_{(l)}^n Z_{(r)j}(\vec{\varepsilon}) \right) \frac{\partial}{\partial \epsilon^n} \\ &+ \frac{1}{m} \left(Z_j^{(n)} \cos(\omega t) + \frac{1}{\omega} \left(\frac{1}{R} \eta_{.jm}^n Z_{(r)}^m \theta^r + \frac{1}{R^2} \delta_j^n \theta_k \epsilon^k \right) \sin(\omega t) \right) \frac{\partial}{\partial \theta^n}. \end{aligned} \quad (32)$$

In spite of the nearly aggressive realization of the new, boost-like, non-point symmetry on the space of evolution of the system, the quantum treatment in terms of the basic symmetries $X_{(i)}, Y_{(j)}$, given by (27), (32), along with their commutators, closing a finite-dimensional group, to be called $SU(2)$ -Sigma group, will be quite cosy. We shall do that in Sec. 3.2.

Let us note that the description above clearly corresponds to that of a particle moving on a space of constant curvature $k = \frac{1}{R^2}$, that is, a sphere S^3 with radius R . Such description generalizes that of the three-dimensional free particle. In fact, if we consider the formal limit $R \rightarrow \infty$, the symmetry generators $X_{(i)}$ (point) and $Y_{(j)}$ (non-point) become those for usual translations and boosts, respectively:

$$\begin{aligned} X_{(i)} &\xrightarrow{R \rightarrow \infty} \frac{\partial}{\partial \epsilon^i} \\ Y_{(j)} &\xrightarrow{R \rightarrow \infty} \frac{\partial}{\partial \dot{\epsilon}^j} + t \frac{\partial}{\partial \epsilon^j}. \end{aligned}$$

This shows again that the generalization of the boosts symmetry for this non-linear case is a non-point symmetry of the Poincaré-Cartan 1-form, although not a symmetry of the Lagrangian.

3 Non-canonical Quantization of $SU(2)$ -PNL σ M

In this Section, we take advantage of the generalized symmetry for the $SU(2)$ particle-non-linear sigma model, describing a free particle moving on a sphere S^3 , just found, to build a proper quantum description. In order to do so, we resort to the Group Approach to Quantization algorithm, outlined in the following Subsection.

3.1 Group Approach to Quantization: a brief review

We give in this section a very brief outline of the so called Group Approach to Quantization (GAQ) or Quantization over a Group Manifold (see [7] and references therein). The basic idea of GAQ consists in taking advantage of having two mutually commuting copies of the Lie algebra $\tilde{\mathcal{G}}$ of a group \tilde{G} which is a central extension of G by $U(1)$ and constitutes the basic, strict symmetry of a given physical system, that is,

$$\chi^L(\tilde{G}) \approx \tilde{\mathcal{G}} \approx \chi^R(\tilde{G}), \quad (33)$$

in such a way that one copy, let us say $\chi^R(\tilde{G})$, plays the role of pre-quantum operators acting (by usual derivation) on complex (wave) functions on \tilde{G} , whereas the other, $\chi^L(\tilde{G})$, is used to reduce the representation in a manner compatible with the action of the operators, thus providing the true quantization.

In fact, from the group law $g'' = g' \star g$ of any group \tilde{G} , we can read two different actions:

$$g'' = g' \star g \equiv L_{g'} g, \quad (34)$$

$$g'' = g' \star g \equiv R_g g'. \quad (35)$$

The two actions commute and so do the generators $\tilde{Z}_{(a)}^R$ and $\tilde{Z}_{(b)}^L$ of the left and right actions respectively, i.e.

$$[\tilde{Z}_{(a)}^L, \tilde{Z}_{(b)}^R] = 0 \quad \forall \text{ group parameters } a, b. \quad (36)$$

The generators $\tilde{Z}_{(a)}^R$ are right-invariant vector fields closing a Lie algebra, $\chi^R(\tilde{G})$, isomorphic to the tangent space to \tilde{G} at the identity, $\tilde{\mathcal{G}}$. The same, changing $L \longleftrightarrow R$, applies to $\tilde{Z}^L \in \chi^L(\tilde{G})$.

We consider the space of complex functions ψ on the whole group \tilde{G} and restrict them to only $U(1)$ -functions, that is, those which are homogeneous of degree one on the argument $\zeta \equiv e^{i\phi} \in U(1)$, that is,

$$\Xi \Psi \equiv \tilde{Z}_{(\phi)}^L \Psi = i \Psi, \quad (37)$$

where Ξ is the (central) generator of the $U(1)$ subgroup. On these functions the right-invariant vector fields act as pre-Quantum Operators by ordinary derivation. However, this action is not irreducible since there is a set of non-trivial operators commuting with this representation. In fact, all the left-invariant vector fields do commute with the right-invariant ones, i.e. the (pre-quantum) operators. According to Schur's Lemma those operators must be "trivialized" in order to turn the right-invariant vector fields into true quantum operators (that is, to achieve an irreducible representation).

We have seen that the action of the central generator (which is, in particular, left-invariant) is fixed to be non-zero by the $U(1)$ -function condition. Thus, not every left-invariant vector field can be nullified in a compatible way with this condition. That is, if

$$[\tilde{Z}_{(a)}^L, \tilde{Z}_{(b)}^L] = \tilde{Z}_{(\phi)}^L, \quad (38)$$

then $\tilde{Z}_{(a)}^L \Psi = 0$, $\tilde{Z}_{(b)}^L \Psi = 0$ is not compatible with $\tilde{Z}_{(\phi)}^L \Psi = i\Psi$. Of course, this null condition on Ψ can be imposed by those generators that never produce a central term by commutation, and they constitute the *Characteristic sub-algebra* \mathcal{G}_Θ . But also half of the rest of left-generators can be joined to \mathcal{G}_Θ to constitute a *Polarization* subalgebra \mathcal{P} .

Then, the role of a polarization is that of reducing the representation, which now constitutes a true *Quantization*. Therefore we impose that wave functions satisfy the polarization condition:

$$\tilde{Z}_{(b)}^L \Psi = 0, \forall \tilde{Z}_{(b)}^L \in \mathcal{P}. \quad (39)$$

The unitarity of the representation is guaranteed by the choice of the invariant volume on the group

$$\mu = \theta^{L(a)} \wedge \theta^{L(b)} \wedge \dots$$

where $\theta^{L(a)}$ are the canonical Left-invariant 1-forms on the group, which obviously is invariant under the quantum operators, rendering them anti-Hermitian (actually Hermitian when multiplied by i).

Particularly useful becomes the canonical 1-form in the central direction, the *quantization* form $\theta^{L(\phi)} \equiv \Theta$, which generalizes the Poincaré-Cartan form of Classical Mechanics, roughly speaking $\Theta \approx \Theta_{PC} + d\phi$. Contrarily to the semi-invariance of Θ_{PC} , the quantization form is strictly invariant. The Characteristic sub-algebra \mathcal{G}_Θ generates the Characteristic Module of Θ , that is $\tilde{Z}/i\tilde{Z}d\Theta = 0 = i\tilde{Z}\Theta$, or generalized equations of motion. Also, the *Noether invariants* find their generalization here as $i\tilde{Z}_{(a)}^R \Theta$.

It should be noticed that the choice of a given polarization subalgebra determines the particular "representation" of the quantum theory, that is, "co-ordinate representation", "momentum representation" or any other. In case the Lie algebra of the quantization group does not provide a polarization subalgebra leading to the desired one, we may resort to the left enveloping algebra to find a higher-order polarization subalgebra of the same dimension and able to accomplish our desire.

3.2 The case of $SU(2)$ -Sigma group

The basic symmetries of a dynamical system are those that close an algebra generalizing the Heisenberg-Weyl algebra of the canonical quantization, in the sense that it contains an even number of operators at least, half of which play the role of "translations" and the other half of "boosts". Obviously, the commutator of "translations" and "boosts" should provide a central term in analogy to the canonical quantization. It is often not easy to determine the basic symmetries of a system and let alone if the system is not linear, as we have realized in Subsec. 2.2. Part of this task can be simplified, as also tested in this example, when working in the solution manifold (SM) of the system [11], while the realization on the evolution space, where the Lagrangian is living, is hard and unpractical.

The quantization of the basic functions on SM will be achieved along the general guides outlined above, particularized for the finite-dimensional Lie group which arises by exponentiating the

Poisson algebra (associated with the symplectic form obtained on SM, (29)) closed by $\varepsilon^i, \vartheta_j$. It is easy to realize the closure of the following 7-dimensional Poisson subalgebra on SM²:

$$\begin{aligned}
\{\varepsilon^i, \varepsilon^j\} &= 0 \\
\{\varepsilon^i, \vartheta_j\} &= \frac{1}{R} \eta_{jk}^i \varepsilon^k + \rho \delta_j^i \\
\{\vartheta_i, \vartheta_j\} &= \frac{2}{R} m \eta_{ij}^k \vartheta_k \\
\{\varepsilon^i, \rho\} &= 0 \\
\{\vartheta_i, \rho\} &= \frac{1}{R^2} \varepsilon_i,
\end{aligned} \tag{41}$$

where the new function ρ , required to close the basic algebra, is

$$\rho(\varepsilon') \equiv \sqrt{1 - \frac{\varepsilon'^2}{R^2}}.$$

The associated Hamiltonian vector fields, now denoted $X_{\vartheta_i}, X_{\varepsilon^j} (= Y^j), X_\rho$, and written in terms of the standard canonical variables, are:

$$\begin{aligned}
X_{\vartheta_i} &= \frac{\partial \vartheta_i}{\partial \pi_s} \frac{\partial}{\partial \varepsilon^s} - \frac{\partial \vartheta_i}{\partial \varepsilon^s} \frac{\partial}{\partial \pi_s} = X_{(i)}^k \frac{\partial}{\partial \varepsilon^k} - \frac{\partial X_{(i)}^k}{\partial \varepsilon^s} \pi_k \frac{\partial}{\partial \pi_s}, \\
X_{\varepsilon^i} &= \frac{\partial \varepsilon^i}{\partial \pi_s} \frac{\partial}{\partial \varepsilon^s} - \frac{\partial \varepsilon^i}{\partial \varepsilon^s} \frac{\partial}{\partial \pi_s} = -\frac{\partial}{\partial \pi_i}, \\
X_\rho &= \frac{\partial \rho}{\partial \pi_s} \frac{\partial}{\partial \varepsilon^s} - \frac{\partial \rho}{\partial \varepsilon^s} \frac{\partial}{\partial \pi_s} = -\frac{1}{R^2 \rho} \varepsilon_s \frac{\partial}{\partial \pi_s}.
\end{aligned}$$

This algebra generalizes and replaces the Heisenberg-Weyl or canonical algebra for the particle over the sphere S^3 and constitutes the basic dynamical symmetry of the system. We insist on the non-point nature of an essential part of this symmetry, a fact that could be responsible of its being unknown, but it is as fundamental as the ordinary Heisenberg-Weyl symmetry for linear systems.

For the sake of completeness, let us mention that the symmetry (41), though being enough to achieve quantization, is not the full symmetry of the system. In fact, it can be added with “ordinary” rotations generated by the Hamiltonian functions (Noether invariants):

$$J_i \equiv \frac{1}{2} \eta_{ijk} \varepsilon^j \pi^k$$

along with the non-independent ones

$$\kappa_i \equiv \rho \pi_i$$

closing now an Euclidean group $E(4)$. In fact, the combinations $J_i + \kappa_i$ and $J_i - \kappa_i$ prove to be the Noether invariants $\theta_i^R \equiv \vartheta_i$ and θ_i^L associated with the “right” and “left” $SU(2)$ generators, leaving invariant the *chiral* Lagrangian (13). This bigger (and non-minimal in the sense that it does not

²The Dirac-bracket method is often used in the treatment of constrained systems, although it is not devoid of ambiguities. In the present case, we might depart from the canonical Poisson bracket associated with the motion of a particle on \mathbb{R}^4 , with global coordinates x^α , $\alpha, \beta = 1, \dots, 4$ subjected to the constraint $x^2 = R^2$. The Dirac brackets would be

$$\{x^\alpha, x^\beta\} = 0, \quad \{x^\alpha, p^\beta\} = \delta^{\alpha\beta} - \frac{x^\alpha x^\beta}{x^2}, \quad \{p^\alpha, p^\beta\} = \frac{1}{x^2} (p^\alpha x^\beta - p^\beta x^\alpha). \tag{40}$$

However, only the coordinates x^i, x^4 , with $i = 1, \dots, 3$, are Noether invariants associated with globally-defined transformations on the solution manifold; the variables p^α must be replaced by global constants like, for instance, $L^i \equiv \frac{1}{2} \eta_{ijk} L^{jk} + L^{i4}$, where $L^{\alpha\beta}$ is the standard angular momentum in four dimensions. In fact, the Lie algebra closed by $\langle L^i, x^j, x^4 \rangle$ is isomorphic to our algebra closed by $\langle \vartheta^i, \varepsilon^j, \rho \rangle$. In addition, ambiguities related with the definition of the quantum Hamiltonian must be solved by hand in the Dirac-bracket method [20].

constitute the minimal generalization of the Heisenberg-Weyl group) symmetry were pointed out in [21, 22] as a possible group to undertake the quantization of the S^3 particle according to the Wigner-Mackey algorithm of induced representations of semi-direct product groups [23]. There, although the realization of this symmetry was not explicit, one would have also been lead to non-point transformations which do not leave the Lagrangian invariant, in contrast with the original aim.

Since we dispose of a well-defined, finite-dimensional group of symmetry we only have to apply step by step the algorithm GAQ just described in order to write down the quantum theory. To this end we exponentiate the algebra above and find the following group law for the $SU(2)$ -sigma group centrally extended by $U(1)$,

$$\begin{aligned}\bar{\varepsilon}'' &= \rho\bar{\varepsilon}' + \rho'\bar{\varepsilon} + \frac{1}{R}\bar{\varepsilon}' \wedge \bar{\varepsilon} \\ \vec{\nu}'' &= \vec{\nu}' + X^L(\bar{\varepsilon}')\vec{\nu} + \frac{1}{R}\bar{\varepsilon}' z \\ z'' &= z' + \rho'z - \frac{1}{R}\bar{\varepsilon}' \cdot \vec{\nu} \\ \zeta'' &= \zeta'\zeta e^{-i\frac{m}{\hbar}(R(\rho'-1)z - \bar{\varepsilon}' \cdot \vec{\nu})},\end{aligned}\tag{42}$$

where $\rho \equiv \rho(\bar{\varepsilon})$, $\rho' \equiv \rho(\bar{\varepsilon}')$, $\zeta = e^{i\phi} \in U(1)$ and the mass $m \in \mathbb{R}$ parameterizes the central extension. The parameters $\vec{\nu}$ and z have the dimensions of a velocity for later convenience. A constant \hbar with the dimensions of an action has to be introduced to keep the exponent dimensionless, although we choose units in which $\hbar = 1$ for simplicity from now on. The left-invariant vector fields of the group are (we shall omit hereafter the symbol \sim over the group generators):

$$\begin{aligned}Z_{(\varepsilon^i)}^L &= Z_{(i)}^L \frac{\partial}{\partial \varepsilon^k} \\ Z_{(\nu^i)}^L &= Z_{(i)}^L \frac{\partial}{\partial \nu^k} - \frac{1}{R}\varepsilon_i \left(\frac{\partial}{\partial z} - mR\Xi \right) \\ Z_{(z)}^L &= \rho \frac{\partial}{\partial z} + \frac{1}{R}\varepsilon^i \frac{\partial}{\partial \nu^i} - mR(\rho - 1)\Xi \\ Z_{(\zeta)}^L &= \Xi = i(\zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}),\end{aligned}\tag{43}$$

where Ξ is the central generator, and the right-invariant ones:

$$\begin{aligned}Z_{(\varepsilon^i)}^R &= Z_{(i)}^R \frac{\partial}{\partial \varepsilon^k} + \frac{1}{R}\eta_{ik}^j \nu^k \frac{\partial}{\partial \nu^j} + \frac{1}{R}z \frac{\partial}{\partial \nu^i} - \frac{1}{R}\nu_i \left(\frac{\partial}{\partial z} - mR\Xi \right) \\ Z_{(\nu^i)}^R &= \frac{\partial}{\partial \nu^i} \\ Z_{(z)}^R &= \frac{\partial}{\partial z} \\ Z_{(\zeta)}^R &= \Xi = i(\zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}).\end{aligned}\tag{44}$$

The Lie algebra (non-null) commutators for the right-invariant vector fields are:

$$\begin{aligned}[Z_{(\varepsilon^i)}^R, Z_{(\varepsilon^j)}^R] &= -\frac{2}{R}\eta_{ij}^k Z_{(\varepsilon^k)}^R \\ [Z_{(\varepsilon^i)}^R, Z_{(\nu^j)}^R] &= -\frac{1}{R}\eta_{ij}^k Z_{(\nu^k)}^R + \frac{1}{R}\delta_{ij} \left(Z_{(z)}^R - mR\Xi \right) \\ [Z_{(\varepsilon^i)}^R, Z_{(z)}^R] &= -\frac{1}{R}Z_{(\nu^i)}^R,\end{aligned}\tag{45}$$

to be compared with (41). The quantization 1-form Θ , dual to Ξ , is

$$\Theta = -m\varepsilon_i d\nu^i - mR(\rho - 1)dz + \frac{d\zeta}{i\zeta},\tag{46}$$

from which we can compute the characteristic sub-algebra

$$\mathcal{G}_\Theta = \langle Z_{(z)}^L \rangle, \quad (47)$$

indicating that the variable z is not symplectic, parametrizing a subgroup which plays a role similar to that of time in an algebraic sense.

Computing the Noether invariants we obtain:

$$\begin{aligned} i_{Z_{(\varepsilon^i)}^R} \Theta &= m \left(Z_{(i)}^{Rk} \nu_k - \frac{1}{R} z \varepsilon_i \right) \\ i_{Z_{(\nu^j)}^R} \Theta &= -m \varepsilon^j \\ i_{Z_{(z)}^R} \Theta &= -mR(\rho - 1) \end{aligned}$$

By taking explicitly quotient by $Z_{(z)}^L$ along with the central generator on the group manifold, we arrive at a symplectic manifold symplectomorphic to the Solution Manifold of the Sec. 2.2 parametrized by the Noether invariants $m\varepsilon^i$ and $m(Z_{(j)}^{Rk} \nu_k - \frac{1}{R} z \varepsilon_j) \equiv m\vartheta_j$, where the notation \equiv indicates the identification of quantities of the quotient taken in the group and quantities in the Lagrangian solution manifold. That quotient manifold in the group, $\tilde{G}/[G_\Theta \otimes U(1)]$, for given m and R , is a symplectic manifold equivalent (symplectomorphic) to the solution manifold of a classical particle of mass m moving on the sphere S^3 of radius R . We refer the reader to Figure 1 in the Appendix to visualize the corresponding quotient and the notation used for the coordinates on each space.

In order to get an irreducible representation it is necessary to impose the polarization conditions, which are

$$\mathcal{P} = \langle Z_{(\nu^i)}^L, Z_{(z)}^L \rangle. \quad (48)$$

After that we arrive at wave functions of the form:

$$\Psi(\zeta, \vec{\varepsilon}, \vec{\nu}, z) = \zeta e^{-im(\vec{\varepsilon} \cdot \vec{\nu} + R(\rho-1)z)} \phi(\vec{\varepsilon}), \quad (49)$$

where $\phi(\vec{\varepsilon})$ is an arbitrary function, save for normalization (see Sec. 3.2.1 later). The quantum operators will be now the right-invariant vector fields. The action on wave functions is

$$\begin{aligned} im\hat{\nu}_i \Psi &\equiv Z_{(\varepsilon^i)}^R \Psi = \zeta e^{-im(\vec{\varepsilon} \cdot \vec{\nu} + R(\rho-1)z)} Z_{(i)}^{Rk} \frac{\partial \phi(\vec{\varepsilon})}{\partial \varepsilon^k} \\ -im\hat{\varepsilon}_i \Psi &\equiv Z_{(\nu^i)}^R \Psi = -im\varepsilon_i \Psi \\ -imR\hat{\rho} \Psi &\equiv Z_{(z)}^R \Psi = -imR(\rho-1) \Psi, \end{aligned} \quad (50)$$

while their action on the wave functions restricted to $\phi(\vec{\varepsilon})$ is

$$\begin{aligned} \hat{\nu}_i \phi(\vec{\varepsilon}) &= -\frac{i}{m} Z_{(i)}^{Rk} \frac{\partial \phi(\vec{\varepsilon})}{\partial \varepsilon^k} \\ \hat{\varepsilon}_i \phi(\vec{\varepsilon}) &= \varepsilon_i \phi(\vec{\varepsilon}) \\ \hat{\rho} \phi(\vec{\varepsilon}) &= (\rho-1) \phi(\vec{\varepsilon}). \end{aligned} \quad (51)$$

Specially relevant becomes the expression of the Hamiltonian. Given that, classically, it is written as $\frac{1}{2m} \delta_{ij} (m\vartheta^i)(m\vartheta^j)$ in terms of the Noether invariants $m\vartheta^i$ (see (24)), the quantum expression is given by replacing classical invariants with the corresponding operators, namely $\hat{\nu}_i$. Therefore, the quantum Hamiltonian is unambiguously defined as $\hat{H} \equiv \frac{1}{2} m \delta^{ij} \hat{\nu}_i \hat{\nu}_j$, which turns out to be the Casimir operator of the $SU(2)$ subgroup:

$$\begin{aligned} \hat{H} \phi(\vec{\varepsilon}) &= -\frac{1}{2m} (Z_{(i)}^{Rk} \frac{\partial Z_{(j)}^{Rm}}{\partial \varepsilon^k} \frac{\partial \phi(\vec{\varepsilon})}{\partial \varepsilon^m} + Z_{(i)}^{Rk} Z_{(j)}^{Rm} \frac{\partial^2 \phi(\vec{\varepsilon})}{\partial \varepsilon^k \partial \varepsilon^m}) \\ &= -\frac{1}{2m} (-\frac{3}{R^2} \varepsilon^m \frac{\partial \phi(\vec{\varepsilon})}{\partial \varepsilon^m} + (\delta^{km} - \frac{\varepsilon^k \varepsilon^m}{R^2}) \frac{\partial^2 \phi(\vec{\varepsilon})}{\partial \varepsilon^k \partial \varepsilon^m}) \\ &\equiv -\frac{1}{2m} \Delta_{L-B} \phi(\vec{\varepsilon}) = E \phi(\vec{\varepsilon}) \end{aligned} \quad (52)$$

where Δ_{L-B} stands for the Laplace-Beltrami operator associated with the metric (17).

For the sake of completeness, we write the expression of the extra operators (corresponding to the invariants $J_i - \kappa_i$, see above) closing the Euclidean group $E(4)$ with (51):

$$ZL_{(\varepsilon^i)}\phi(\vec{\varepsilon}) = Z_{(i)}^{Lk} \frac{\partial \phi(\vec{\varepsilon})}{\partial \varepsilon^k} \quad (53)$$

They also close the $SO(4)$ subgroup of $E(4)$ with $ZR_{(\varepsilon^i)} \equiv Z_{(\varepsilon^i)}^R$ associated with $J_i + \kappa_i$. As a consequence, the combination

$$\hat{J}_i \equiv \frac{R}{2}(ZR_{(\varepsilon^i)} - ZL_{(\varepsilon^i)}) = \eta_i^{jk} \varepsilon_j \frac{\partial}{\partial \varepsilon^k} \quad (54)$$

is in the Lie algebra of the group $E(4)$ and is obviously interpreted as the generators of usual rotations on the S^3 space.

At this point, it is interesting to consider once more the formal limit $R \rightarrow \infty$ in the quantum operators obtained. With the expression of $Z_{(i)}^{Rk}$ and $Z_{(i)}^{Lk}$ in mind (see (18) and (19)), it is straightforward to check that $\hat{\nu}_i$ goes to the usual momentum operator (over m), $\hat{\varepsilon}_i$ goes to the usual position operator ($\vec{\varepsilon}$ is not bounded in the limit) and the Hamiltonian \hat{H} goes to the usual three-dimensional free particle Hamiltonian operator, provided the appropriate identification of coordinates is made. Also, rotations (54) are preserved in the limit.

From an algebraic point of view, this should not be surprising: it can be shown that there exists an Inönü-Wigner contraction [24] of Lie algebra (45), corresponding to the limit $R \rightarrow \infty$, leading to the Heisenberg-Weyl algebra of the canonical quantization $[\hat{x}, \hat{p}] = i$ added with the operator \hat{x}^2 and rotations, i.e., a subalgebra of the Schrödinger algebra of symmetries of the quantum free particle which does not include the time evolution symmetry nor dilations. Only after that contraction, \hat{H} closes a Lie algebra with basic operators, namely, the Schrödinger algebra.

From the previous discussion, it becomes patent again that the construction above of the quantum theory for the free particle on S^3 is a natural generalization of that for the free Galilean particle. Let us stress that having found the relevant non-point symmetries of the classical system is essential for such generalization.

3.2.1 Integration measure: Hilbert space

In order to achieve a complete quantum description of the system, it is necessary to define a measure on the Hilbert space. For functions on a Lie group, there is a canonical integration measure: the Haar measure ω . The Haar measure for the $SU(2)$ -Sigma group is given by

$$\omega = \theta^L(\varepsilon_1) \wedge \theta^L(\varepsilon_2) \wedge \theta^L(\varepsilon_3) \wedge \theta^L(\theta_1) \wedge \theta^L(\theta_2) \wedge \theta^L(\theta_3) \wedge \theta^L(z). \quad (55)$$

On polarized wave functions, in the context of GAQ, it is possible to define an invariant measure [25] as follows

$$d\mu = i_{X_{\theta^1}} i_{X_{\theta^2}} i_{X_{\theta^3}} i_{X_z} \omega = \theta^L(\varepsilon_1) \wedge \theta^L(\varepsilon_2) \wedge \theta^L(\varepsilon_3) = \frac{1}{\sqrt{1 - \frac{1}{R^2} \vec{\varepsilon} \cdot \vec{\varepsilon}}} d\varepsilon^1 \wedge d\varepsilon^2 \wedge d\varepsilon^3 \quad (56)$$

$$\equiv \sqrt{|g|} d\varepsilon^1 \wedge d\varepsilon^2 \wedge d\varepsilon^3. \quad (57)$$

This measure is the Haar measure on the $SU(2)$ group as expected, as well as the standard measure on a Riemannian manifold (S^3) with metric g and determinant $|g|$. The volume on the whole group, as is known, is

$$\int_{S^3} d\mu = 2\pi^2 R^3. \quad (58)$$

Finally, the scalar product between two wave functions restricted to configuration space has the following expression

$$\langle \Psi' | \Psi \rangle = \int_{S^3} \overline{\phi'(\vec{\varepsilon})} \phi(\vec{\varepsilon}) d\mu. \quad (59)$$

All the quantum operators found in Sec. 3.2 turn out to be self-adjoint with respect to this scalar product and, therefore, the representation is unitary.

3.2.2 Solutions

We now arrive at the search for a definite basis for the (Hilbert) space of wave functions, carrying a unitary and irreducible representation of our symmetry group characterizing the quantum dynamics of a particle moving on S^3 . As usual, we choose eigenfunctions of \hat{H} , but two more operators must be simultaneously diagonalized in order to resolve the degeneracy. A possibility (not unique) is the choice $\langle \hat{H}, \hat{J}^2, \hat{J}_3 \rangle$, with $\hat{\vec{J}} = \frac{R}{2}(ZR_{(\vec{\varepsilon})} - ZL_{(\vec{\varepsilon})})$ as before, as a maximal set of commuting observables.

To achieve the actual construction of the wave functions basis, it is convenient to resort to a hyperspherical coordinate system in the form:

$$\begin{aligned}\varepsilon_1 &= R \sin \chi \sin \theta \cos \phi \\ \varepsilon_2 &= R \sin \chi \sin \theta \sin \phi \\ \varepsilon_3 &= R \sin \chi \cos \theta \\ (\rho &= \cos \chi),\end{aligned}\tag{60}$$

where the χ variable completes the ordinary spherical coordinates (θ, ϕ) . In those variables, the required eigen-problem can be solved with the result:

$$\psi_{nlm}(\chi, \theta, \phi) = N_{nlm} \sin^l \chi C_{n-l}^{(l+1)}(\cos \chi) Y_{lm}(\theta, \phi),\tag{61}$$

where $C_{n-l}^{(l+1)}(x)$ are the Gegenbauer polynomials in the x variable, $Y_{lm}(\theta, \phi)$ are the ordinary spherical harmonics, and N_{nlm} are the following normalizing constants:

$$N_{nlm} = 2^l l! \sqrt{\frac{2(n+1)(n-l)!}{\nu(n+l+1)!}}.\tag{62}$$

The wave functions solve the eigen-problem according to the expressions:

$$\begin{aligned}\hat{H}\psi_{nlm} &= \frac{n(n+2)}{2mR^2}\psi_{nlm} \\ \hat{J}^2\psi_{nlm} &= l(l+1)\psi_{nlm} \\ \hat{J}_3\psi_{nlm} &= m\psi_{nlm}.\end{aligned}$$

3.2.3 “Momentum-space” quantization

So far we have realized the quantization of the S^3 -sigma particle in say the “configuration space” or “coordinate representation”, since the variables upon which the wave functions depend arbitrarily are the parameters ε . The “momentum space” can also be achieved from our quantization group by looking for a polarization subalgebra containing the generator $X_{\varepsilon^i}^L$. Unfortunately, no first-order polarization subalgebra does exist containing this generator and we have to seek in the left-enveloping algebra. In fact, we can construct the following higher-order polarization subalgebra:

$$\mathcal{P}^{HO} = \langle Z_{(\vec{\varepsilon}^i)}^L, Z_{(z)}^{LHO} \rangle\tag{63}$$

where the higher-order left generator replacing $Z_{(z)}^L$ is

$$Z_{(z)}^{LHO} \equiv \left(Z_{(z)}^L \right)^2 - 2imR Z_{(z)}^L + Z_{(\vec{\nu})}^L \cdot Z_{(\vec{\nu})}^L,\tag{64}$$

and commutes with $Z_{(\vec{\varepsilon}^i)}^L$. The polarization conditions on the $U(1)$ -wave functions $\zeta\psi(z, \vec{\varepsilon}, \vec{\nu})$ now start by imposing the first-order condition:

$$Z_{(\vec{\varepsilon})}^L \psi(z, \vec{\varepsilon}, \vec{\nu}) = 0 \Rightarrow \psi \neq \psi(\vec{\varepsilon}).\tag{65}$$

The condition of second order acquires a simple form after realizing that $Z_{(z)}^{LHO}$ does commute with the entire algebra and thus can be rewritten as $Z_{(z)}^{RHO}$, that is, using the same algebraic expression though in terms of right-invariant generators. We obtain:

$$\left[-2imR \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \vec{v}^2} \right] \psi = 0. \quad (66)$$

With a simple redefinition of the wave function,

$$\psi(z, \vec{v}) = e^{imRz} \phi(z, \vec{v}), \quad (67)$$

intended to subtract the additive constant i accompanying $\frac{\partial}{\partial z}$, we arrive at

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \vec{v}^2} + m^2 R^2 \right] \phi(z, \vec{v}) = 0 \quad (68)$$

the solutions of which are eigen-functions of the ‘‘Laplacian’’ operator $\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \vec{v}^2}$.

The solution of the polarization equations and the reduction of the representation to the space of functions depending only on \vec{v} (momentum) deserves further study and will be presented elsewhere.

4 Outlook and final remarks

In this article we have realized a consistent quantization of a particle moving on the sphere S^3 considered as the parameter space of the group $SU(2)$. It constitutes a paradigmatic yet relatively simple non-linear (quantum-Mechanical) sigma-model problem. To realize the proper quantization we have resorted to a (non-canonical) Group Approach to Quantization, entirely based on symmetry grounds and generalizing the Kostant-Kirillov-Souriau technique for Lie group representation. The minimal, basic symmetry to achieve this task turned out to be constituted, in part, by symmetries of the Poincaré-Cartan form that do not preserve the Lagrangian, that is, non-point symmetries; this sort of symmetries are rather well known though mildly used in quantum theory. The use of Lie group techniques in the quantization process of non-linear classical systems solves the order ambiguities which inescapably arise in the Canonical Quantization scheme, provides the adequate integration measure on non-trivial configuration spaces and a generalization of the momentum-space ‘‘representation’’, which requires further analysis and will be studied elsewhere. The ordinary Canonical Quantization process naturally emerges also as group quantization in the Inönü-Wigner contraction (radius $\rightarrow \infty$) that turns, roughly speaking, the $SU(2)$ -sigma group into the Heisenberg-Weyl one.

Even though the classical system is exactly solved, we have proceeded in a more practical way consisting in quantizing the solution manifold (SM) as associated with the $SU(2)$ -sigma group (as a co-adjoint orbit) and introducing then the Hamiltonian operator as an unambiguous function of the already represented basic generators of the symmetry group.

We should mention that the present quantized system can also serve as a toy model for future proper quantization of $SU(2)$ -sigma models in field theory. The ‘‘local’’ $SU(2)$ -sigma group was already presented, at the Lie algebra level, in a previous paper [11] and it would be intended to provide the adequate treatment of the bosonic sector in Non-Abelian Stueckelberg (massive-gauge) theories of interactions.

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Appendix

We provide here a diagram in which the notation for coordinates in each space is specified. Note that the diagram refers to the classical description, although the notation for the group is maintained for the quantum description.

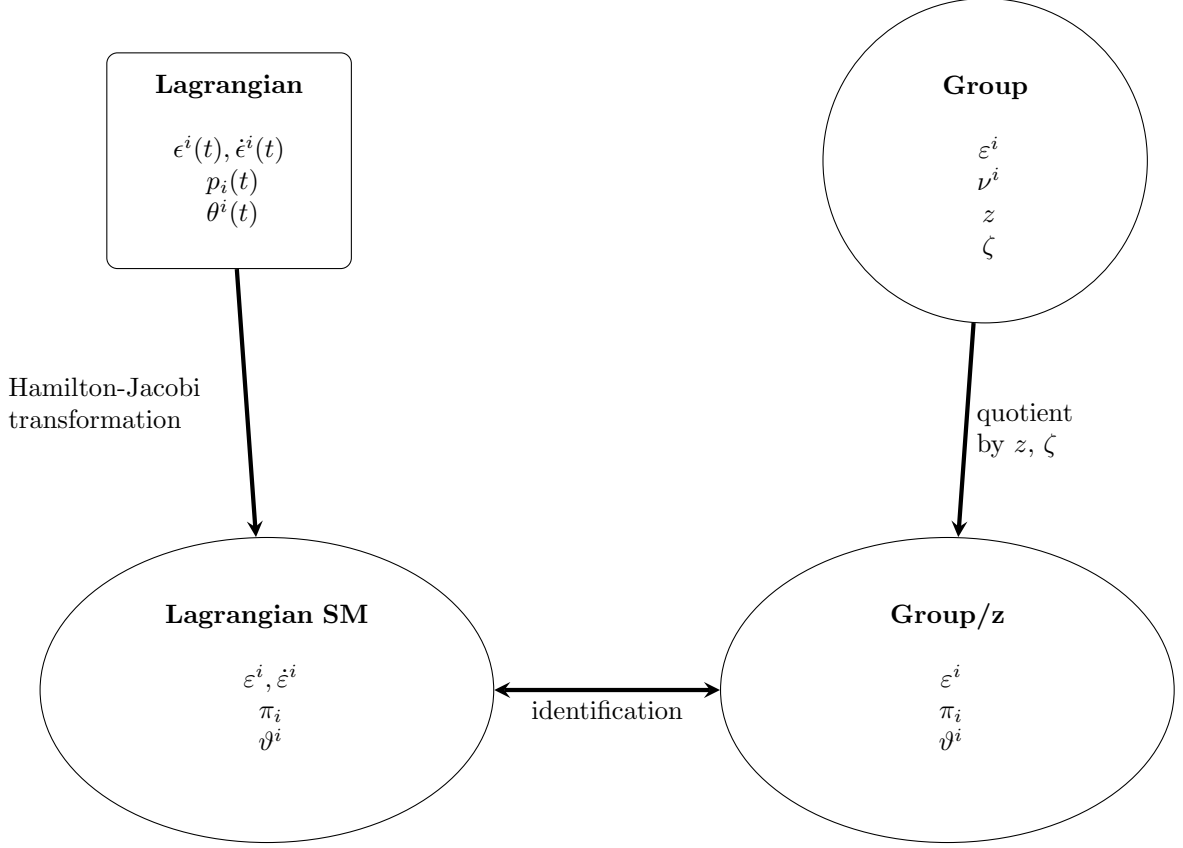


Figure 1: Notation for coordinates in Lagrangian mechanics and Group Manifolds

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